

Cobb–Douglas

Consumer theory · Problem 3 · Video: youtu.be/SsDJB3uKrfw

Solved problem

A student splits a monthly allowance between food and outings. Writing

- x : food, at price p_x ;
- y : outings, at price p_y ,

tastes are Cobb–Douglas, $u(x, y) = x^\alpha y^\beta$ with $\alpha, \beta > 0$. With a budget of M dollars, find the demand functions $x^d(p_x, p_y, M)$ and $y^d(p_x, p_y, M)$. Then illustrate the choice for $\alpha = 2$, $\beta = 1$, $p_x = p_y = 2$ and a \$12 budget.

Solution

The student solves

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}_{++}^2} \quad & x^\alpha y^\beta \\ \text{s.t.} \quad & p_x x + p_y y \leq M. \end{aligned}$$

The student wants some of both goods, so the best bundle is interior. There the indifference curve just touches the budget line, which means the marginal rate of substitution equals the price ratio:

$$\text{MRS} = \frac{\text{MU}_x}{\text{MU}_y} = \frac{\alpha x^{\alpha-1} y^\beta}{\beta x^\alpha y^{\beta-1}} = \frac{\alpha y}{\beta x} = \frac{p_x}{p_y}.$$

Rearranged, this says $\beta p_x x = \alpha p_y y$: the money spent on food and the money spent on outings stand in the ratio $\alpha : \beta$. The budget binds, $p_x x + p_y y = M$, so each share is a fixed slice of the budget,

$$p_x x^d = \frac{\alpha}{\alpha + \beta} M, \quad p_y y^d = \frac{\beta}{\alpha + \beta} M,$$

which gives the demand functions

$$x^d(p_x, p_y, M) = \frac{\alpha}{\alpha + \beta} \frac{M}{p_x}, \quad y^d(p_x, p_y, M) = \frac{\beta}{\alpha + \beta} \frac{M}{p_y}.$$

So the student always spends the constant fractions $\frac{\alpha}{\alpha + \beta}$ and $\frac{\beta}{\alpha + \beta}$ of the budget on the two goods, whatever the prices or income are.

With $\alpha = 2$, $\beta = 1$, $p_x = p_y = 2$ and $M = 12$ the shares are $\frac{2}{3}$ and $\frac{1}{3}$, so

$$x^d(2, 2, 12) = \frac{2}{3} \cdot \frac{12}{2} = 4, \quad y^d(2, 2, 12) = \frac{1}{3} \cdot \frac{12}{2} = 2,$$

that is $(x^*, y^*) = (4, 2)$: \$8 on food and \$4 on outings.

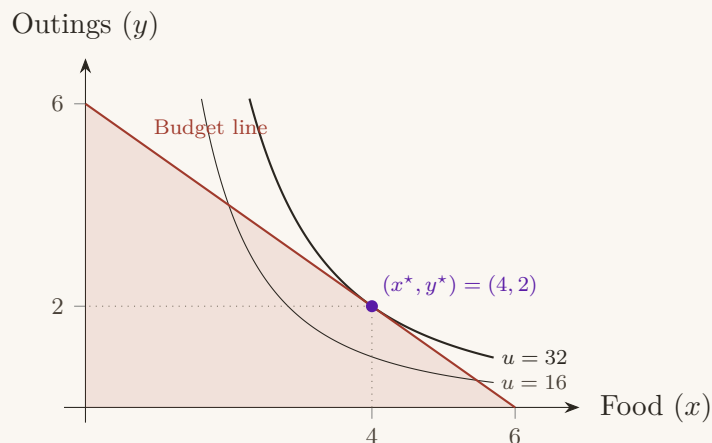


Figure 1. With Cobb–Douglas tastes the indifference curves are smooth and convex. The student climbs to the highest curve the budget allows, where it just touches the budget line. At that tangency the spending on each good is a fixed fraction of the budget — here $\frac{2}{3}$ on food and $\frac{1}{3}$ on outings — giving $(x^*, y^*) = (4, 2)$.

Exercise

A household buys grain and vegetables. It needs a guaranteed minimum of each before anything else — γ_x of grain and γ_y of vegetables — and beyond those floors its tastes are Cobb–Douglas. Writing

- x : grain, at price p_x ;
- y : vegetables, at price p_y ,

the household’s tastes are Stone–Geary,

$$u(x, y) = (x - \gamma_x)^\alpha (y - \gamma_y)^\beta, \quad \alpha, \beta > 0, \quad \gamma_x, \gamma_y \geq 0,$$

defined for $x \geq \gamma_x$ and $y \geq \gamma_y$. The budget is M dollars, with $M > p_x \gamma_x + p_y \gamma_y$ so that the floors are affordable.

1. Find the demand functions $x^d(p_x, p_y, M)$ and $y^d(p_x, p_y, M)$. *Hint:* let $\tilde{x} = x - \gamma_x$ and $\tilde{y} = y - \gamma_y$; the problem becomes Cobb–Douglas in \tilde{x}, \tilde{y} with the leftover budget $M - p_x \gamma_x - p_y \gamma_y$.
2. Take $\alpha = 2$, $\beta = 1$, $\gamma_x = 1$, $\gamma_y = 2$, $p_x = p_y = 2$ and $M = 12$. Find the optimal bundle, and compare it with the Cobb–Douglas answer $(4, 2)$ from the solved problem.

Extension: shares are no longer constant

3. Using your demand function x^d from part 1, show that a one-dollar rise in income M raises spending on grain by the fixed amount $\frac{\alpha}{\alpha + \beta}$, yet the *share* of total spending that goes to grain still changes with M . Find that share as a function of M and show it approaches $\frac{\alpha}{\alpha + \beta}$ as $M \rightarrow \infty$.

Checking your work. To discuss the exercise, join the community forum at econschool.in.